

# Spinor class fields for sheaves of lattices

Luis Arenas-Carmona\*  
 Universidad de Chile,  
 Facultad de Ciencias,  
 Casilla 653, Santiago, Chile  
 E-mail: learenas@uchile.cl

April 12, 2011

## Abstract

We extend the theory of spinor class field and representation fields to any linear algebraic group over a global function field satisfying some technical conditions that ensure the existence of a suitable spinor norm. This is the analog of a result given by the author in the number field case. Spinor class fields can still be defined for lattices defined over a projective curve in a sheaf-theoretical context. Spinor genus is a rather weak invariant in this context, by it can be used to study the behavior of the genus at affine subsets. Examples are provided.

## 1 Introduction

Let  $n$  be a positive integer and let  $k$  be a number field with ring of integers  $\mathcal{O}_k$ . Every conjugacy class of maximal orders in the matrix algebra  $\mathbb{M}_n(k)$  has a representative of the form  $\mathfrak{D}_n(I) = (I^{\delta_{1,i} - \delta_{1,j}})_{i,j}$ , where  $(I_{i,j})_{i,j}$  is the lattice of matrices  $(a_{i,j})_{i,j}$  satisfying  $a_{i,j} \in I_{i,j}$  for ideals  $I_{i,j}$  ([9], p. 18), i.e.,

$$\mathfrak{D}_2(I) = \begin{pmatrix} \mathcal{O}_k & I \\ I^{-1} & \mathcal{O}_k \end{pmatrix}, \mathfrak{D}_3(I) = \begin{pmatrix} \mathcal{O}_k & I & I \\ I^{-1} & \mathcal{O}_k & \mathcal{O}_k \\ I^{-1} & \mathcal{O}_k & \mathcal{O}_k \end{pmatrix}, \dots$$

---

\*Supported by Fondecyt, proyecto No. 1085017.

Two such orders  $\mathfrak{D}_n(I)$  and  $\mathfrak{D}_n(I')$  are conjugate if and only if  $J^n I' I^{-1}$  is principal for some fractional ideal  $J$  of  $k$  ([9], p. 23). It follows that the set of conjugacy classes of maximal orders in  $\mathbb{M}_n(k)$  is in correspondence with the quotient  $\mathfrak{g}/\mathfrak{g}^n$  of the class group  $\mathfrak{g}$  of  $k$ . In [4] we extended this theory to the case where  $k$  is a global function field and  $\mathcal{O}_k$  is an arbitrary Dedekind domain whose field of quotients is  $k$ . We can even consider the projective case if we replace the ring  $\mathcal{O}_k$  by the structure sheaf  $\mathcal{O}_X$  of a smooth projective curve  $X$  whose field of rational functions is  $k$ , and we interpret maximal orders and ideals in the preceding statements in a sheaf-theoretical setting. In that context, it can only be proved that every spinor genera of maximal orders can have a representative of the form  $\mathfrak{D}_n(I)$ . In fact, the existence of non-split maximal orders in  $\frac{1}{n}$ -th of all spinor genera is a consequence of the Theory of representation by spinor genera (§4). In this article we extend this theory to any group having an appropriate spinor norm, as it was done for the number field case in [1].

Let  $X$  be a smooth projective curve over a finite field  $\mathbb{F}$ . Let  $K(X)$  be the field of rational functions on  $X$ , and let  $G$  be an algebraic subgroup of  $\text{Aut}_{K(X)} V$ , for some  $K(X)$ -vector space  $V$ . Let  $C$  be an arbitrary non-empty Zariski-open subset of  $X$ . We allow the case  $C = X$ . Let  $\Lambda$  and  $M$  be  $C$ -lattices on  $V$ . We say that  $\Lambda$  and  $M$  are:

1. in the same  $G$ -class if  $g(\Lambda) = M$  for some  $g \in G$ ,
2. in the same  $G$ -genus if for every point  $\wp \in C$  there exists an element  $g_\wp \in G_\wp$  such that  $g_\wp(\Lambda_\wp) = M_\wp$ , where  $\Lambda_\wp$  and  $M_\wp$  denote the completions at  $\wp$ ,
3. in the same  $G$ -spinor genus if there exists a lattice  $P$  satisfying the following conditions:
  - (a) There exists an element  $h \in G$  such that  $h(\Lambda) = P$ ,
  - (b) For every point  $\wp \in C$  there exists an element  $g_\wp \in G_\wp$  with trivial spinor norm (§2) such that  $g_\wp(P_\wp) = M_\wp$ .

Spinor genera and classes coincide whenever  $G$  is non-compact at some infinite point of  $C$  if any (Proposition 7). In §3 we prove the following result:

**Theorem 1.** *Let  $G$  be any semi-simple linear algebraic group  $G$ , defined over the field of functions  $K = K(X)$  of a smooth projective curve  $X$ . Assume  $G$*

satisfies the technical conditions **SN** and **RU** in §2 below. Then the set of  $G$ -spinor genera in the  $G$ -genus of any  $C$ -lattice  $\Lambda$  is a principal homogeneous space over the Galois group  $\mathcal{G}$  of an Abelian extension  $\Sigma_\Lambda^C/K(X)$  called the spinor class field of the lattice  $\Lambda$ . This extension splits completely at every infinite point of  $C$  if any. If  $D \subseteq C$  is open and  $M$  is the restriction of  $\lambda$  to  $D$  as a sheaf (or equivalently the  $\mathcal{O}_X(D)$ -lattice generated by  $\Lambda$ ), then  $\Sigma_\Lambda^C/K(X)$  is the maximal subextension of  $\Sigma_\Lambda^C/K(X)$  splitting at every place in  $C \setminus D$ .

In other words, we have a natural action of  $\mathcal{G}$  on the set  $\Omega$  of spinor genera in the genus, and for every pair of lattices  $(M, N)$  there exists a unique element  $\rho(M, N) \in \mathcal{G}$  taking the spinor genus of  $M$  to the spinor genus of  $N$ . The map  $\rho$  satisfies the relation  $\rho(M, P) = \rho(M, N)\rho(N, P)$  for any three lattices  $M, N$ , and  $P$  in the genus. The class field  $\Sigma_\Lambda^C$  depends only on the genus of the lattice  $\Lambda$ .

**Example A.** Assume  $q$  is odd. Let  $\{C_i\}_{i=1}^m$  be an affine cover of an irreducible smooth projective curve  $X$  over  $\mathbb{F}_q$ . For  $i = 1, \dots, m$ , let  $\Lambda_i$  be a  $C_i$ -lattice in a fixed regular quadratic or skew-hermitian  $K(X)$ -space  $W$ . Assume that the completions  $\Lambda_{i,\wp}$  and  $\Lambda_{j,\wp}$  coincide whenever  $\wp \in C_i \cap C_j$ . Then there exists a class field  $\Sigma$  such that, for every affine subset  $D$  of  $X$ , if  $M$  is the  $D$ -lattice satisfying  $M_\wp = \Lambda_{i,\wp}$  for  $\wp \in C_i \cap D$ , then the spinor class field  $\Sigma_M^D$  is the maximal subfield of  $\Sigma$  splitting completely at the infinite places of  $D$ . We simply define  $\Sigma$  as the spinor class field of the lattice obtained by pasting together the lattices  $\Lambda_i$  as sheaves.

It is apparent that  $\Sigma \supseteq \Sigma_{\Lambda_1}^{C_1} \cdots \Sigma_{\Lambda_m}^{C_m}$ , but equality does not need to hold. For example, assume  $X = \mathbb{P}^1(\mathbb{F}_q) = C_1 \cup C_2$  with either  $C_i$  affine. Let  $\Lambda$  be a free  $X$ -lattice with a basis  $v_1, \dots, v_n$  for some  $n \geq 3$ , and let  $\Lambda_i$  be the restriction of  $\Lambda$  to  $C_i$ , for  $i = 1, 2$ . Let  $Q$  be the quadratic form defined by

$$Q\left(\sum_{i=1}^n g_i v_i\right) = \sum_{i=1}^{n-2} g_i^2 + g_{n-1}g_n.$$

Then every spinor genus has a representative of the form

$$L(I) = \bigoplus_{i=1}^{n-2} \mathcal{O}_C v_i \perp (I v_{n-1} \oplus I^{-1} v_n),$$

for a suitable ideal  $I$ . This shows that  $\Sigma_{\Lambda_i}^{C_i} = K(X)$  for either affine set  $C_i$ , but we claim that  $\Sigma$  must contain a quadratic extension. The claim follows by considering an open subset  $D$  isomorphic to the rational affine curve with equation  $y = P(x, y)$  where  $P$  is an irreducible quadratic form on  $\mathbb{F}_q[x, y]$ . It suffices to prove that the class group of the ring  $\mathcal{O}_X(D)$  has even order. In fact, the ideal  $I = (x, y)$  is not principal, but  $I^2 = (y)$ , as an easy computation shows.

In fact, the spinor genus of the lattice  $L(I)$  can be computed from the divisor class defining the ideal  $I$ . We show in §4 that not every class of lattices in the genus of  $L(\mathcal{O}_X)$  has a representative of this type. We do this by extending the theory of representation fields to the sheaf setting. When strong approximation fails, as is always the case for  $X$ -lattices, representation fields give only information on the number of spinor genera representing a given lattice. Just as in the number field case, representation fields might fail to exist, but they do exist in many important families of examples.

## 2 Spinor norms

To fix ideas, let  $X$  be the irreducible smooth projective algebraic curve defined over a finite field  $\mathbb{F} = \mathbb{F}_q$ , and let  $K(X)$  be its field of rational functions. We say that a semi-simple linear algebraic group  $G \subseteq \mathrm{GL}(n, K(X))$ , defined over  $K(X)$ , satisfies condition **SN** if:

1. *The extension of the universal cover  $\phi : \tilde{G} \rightarrow G$  of  $G$  to any separably closed field  $E$  containing  $K(X)$  is surjective.*
2. *For almost all points  $\wp \in X$ , any integral element  $g$  of the completion  $G_\wp$  has a pre-image in  $\tilde{G}_Z$  for some unramified extension  $Z/K_\wp$ .*

Let  $F = \ker \phi$  be the fundamental group of a semi-simple group  $G$  satisfying condition **SN**. The cohomology function  $\theta : G \rightarrow H^1(K(X), F_E)$  arising from the short exact sequence  $F_E \hookrightarrow \tilde{G}_E \twoheadrightarrow G_E$ , where  $E$  is the separable closure of  $K(X)$ , is called the spinor norm. It is also defined at any field containing  $K(X)$ . The spinor norm on a completion  $K_\wp$  of  $K(X)$  is denoted  $\theta_\wp$ . There exists also an adelic version of the spinor norm. It is the map

$$\Theta : G_{\mathbb{A}} \rightarrow \prod_{\wp \in X} H(K_\wp, F), \quad \Theta(g) = \left( \theta_\wp(g_\wp) \right)_\wp,$$

where  $G_{\mathbb{A}} \subseteq \prod_{\wp} G_{\wp}$  is the adelization of the group  $G$  [12].

**Lemma 1.** *For any semi-simple group  $G$  satisfying condition **SN** the spinor norm is surjective over  $K(X)$  and over any localization  $K_{\wp}$ .*

*Proof.* Recall that  $H^1(K, \tilde{G}) = \{1\}$  for both, the global field  $K = K(X)$  and the local field  $K = K_{\wp}$  [17]. Now the result follows by applying cohomology to the short exact sequence  $F_E \hookrightarrow \tilde{G}_E \twoheadrightarrow G_E$ , where  $E$  is the separable closure of  $K(X)$  or  $K_{\wp}$ .  $\square$

Let  $E$  be the separable closure of  $K(X)$ . In what follows, we say that  $G$  satisfies condition **RU** if it satisfies the following conditions:

1. *The fundamental group  $F_E$  of  $G_E$  is cyclic, and its order  $n$  is not divisible by the characteristic of  $\mathbb{F}$ . In particular, it is isomorphic to the group  $\mu_n$  of  $n$ -roots of unity in  $E$ .*
2. *There exists an isomorphism between  $F_E$  and  $\mu_n$  commuting with the natural action of the Galois group  $\text{Gal}(E/K(X))$  on either group.*

Condition **RU** implies that  $H^1(K(X), F_E) \cong K(X)^{*n}/K(X)^*$ . Note that if  $E(\wp)$  is the separable closure of  $K_{\wp}$ , then  $F_{E(\wp)} = F_E$ , since  $K_{\wp}$  contains no inseparable extensions of  $K(X)$  ([18], §VIII.6.). It follows that  $H^1(K_{\wp}, F_E) \cong K_{\wp}^{*n}/K_{\wp}^*$  for any  $\wp \in X$ .

**Lemma 2.** *For any semi-simple group  $G$  satisfying conditions **SN** and **RU** the image of the adelic spinor norm  $\Theta$  is contained in  $J_X/J_X^n$ , where  $J_X$  is the idele group of  $K(X)$ .*

*Proof.* Observe that for any extension  $E/K_{\wp}$  such that a given element  $g \in G_{K_{\wp}} = G_{\wp}$  is in the image of the cover  $\phi : \tilde{G}_E \rightarrow G_E$ , the spinor norm  $\theta_{\wp}(g)$  can be computed from the short exact sequence  $F_E \hookrightarrow \tilde{G}_E \twoheadrightarrow \phi(\tilde{G}_E)$ , whence it lies in  $H^1(E/K_{\wp}, F_E) \cong (E^{*n} \cap K_{\wp}^*)/K_{\wp}^{*n}$ . The latter isomorphism is just the coboundary map of the short exact sequence  $F_E \hookrightarrow E^* \twoheadrightarrow E^{*n}$ .

By condition **SN** every  $g \in G_{\mathcal{O}_{\wp}}$  is the image of some  $h \in \tilde{G}_E$  for some unramified extension  $E/K_{\wp}$ , for almost all points  $\wp \in X$ . This implies

$$\theta_{\wp}(g) \in (E^{*n} \cap K_{\wp}^*)/K_{\wp}^{*n} \subseteq \mathcal{O}_{\wp}^* K_{\wp}^{*n}/K_{\wp}^{*n},$$

whence  $\theta_{\wp}(G_{\mathcal{O}_{\wp}}) \subseteq \mathcal{O}_{\wp}^* K_{\wp}^{*n}/K_{\wp}^{*n}$  for almost all  $\wp$ .  $\square$

The automorphism groups of the structures mentioned in the introduction indeed satisfy these conditions.

**Lemma 3.** *Orthogonal groups of regular quadratic forms and unitary groups of regular quaternionic skew-hermitian forms satisfy conditions **SN** and **RU** if the characteristic of the base field  $\mathbb{F}$  is not 2.*

*Proof.* The universal cover of the Orthogonal group of a quadratic space  $(V, q)$  over a field  $K$  whose characteristic is not 2 is the spin group  $\text{Spin}(q)$ . It is defined as the set of elements  $u$  in the Even Clifford Algebra  $C^+(q)$  [11] satisfying  $u\bar{u} = 1$  and  $uVu^{-1} = V$ . Condition **RU** follows since any element  $u$  in the spin group satisfying  $uvu^{-1} = v$  for any  $v \in V$  is in the base field  $K(X)$  ([11], §54:4), whence  $u\bar{u} = 1$  implies  $u = \pm 1$ .

If the field  $E \supseteq K$  is separably closed, then every product of two symmetries  $\tau_v \tau_w$  is the image of  $\frac{vw}{\sqrt{q(v)q(w)}} \in \text{Spin}(q)_E$ . These elements generate the special orthogonal group [11]. Furthermore, if  $K = K_\wp$  is a local field and  $q$  is a unimodular integral quadratic form at  $\wp$ , the integral orthogonal group is generated by products of 2 reflections  $\tau_v \tau_w$  where  $q(v)$  and  $q(w)$  are units ([11], §92:4), whence  $\frac{vw}{\sqrt{q(v)q(w)}}$  is defined over an unramified extension.

For unitary groups of quaternionic skew-hermitian forms the proof follows from the previous case, since any quaternion algebra splits on some separable quadratic extension of  $K(X)$  ([14], Thm. 7.15), which ramifies at only finitely many places (Theorem 1 in §VIII.4 of [18]), and the unitary group of a skew-hermitian form on a split quaternion algebra is isomorphic to an orthogonal group ([7], Lemma 3).  $\square$

A similar result can be proved for the automorphism group of a central simple algebra of dimension  $n^2$  when  $n$  is not divisible by the characteristic of  $\mathbb{F}$ . However, we have a stronger result:

**Lemma 4.** *If  $G$  is the automorphism group of a central simple algebra  $\mathfrak{A}$ , the reduced norm map  $\Theta = N : G_{\mathbb{A}} \rightarrow J_X/J_X^n$  satisfies the conclusions of Lemma 2 and Lemma 1 regardless of the characteristic.*

*Proof.* See ([18], §X.2, Prop 6) and ([18], §XI.3, Prop 3) for Lemma 1.

Passing to a separable extension if needed we assume that  $\mathfrak{A}$  is isomorphic to a matrix algebra  $\mathbb{M}_t(K(X))$ . Restricting to a smaller set of points  $\wp$  if needed, we assume that the isomorphism maps the standard basis of the

matrix algebra to a basis of the lattice of integral elements in  $\mathfrak{A}$ . Then the integral elements of  $G$  are just the automorphisms of  $\mathbb{M}_t(K_\varphi)$  fixing  $\mathbb{M}_t(\mathcal{O}_\varphi)$ , i.e.  $PGL(n, \mathcal{O}_\varphi)$ . Any element  $g \in PGL(n, \mathcal{O}_\varphi)$  has reduced norm in  $\mathcal{O}_\varphi^* K_\varphi^{*n}$  and the conclusion of Lemma 2 follows.  $\square$

We define the spinor norm for the automorphism group of an algebra  $\mathfrak{A}$  as the reduced norm as above. When  $\text{char}(\mathbb{F})$  divides  $n$ , the map  $SL_1(\mathfrak{A}_E) \rightarrow \text{Aut}_E \mathfrak{A}_E$  fails to be surjective, so we cannot interpret the spinor norm as a co-boundary. However, the explicit construction of  $\Theta$  is not used in the remaining of this work.

### 3 $X$ -lattices

In this section we recall the properties of  $X$ -lattices that are used in the sequel. Let  $X$  be a smooth irreducible projective curve over a finite field  $\mathbb{F}$ . Let  $K = K(X)$  be the field of rational functions on  $X$ , and for every place  $\varphi \in X$  we let  $K_\varphi$  be the completion at  $\varphi$  of  $K$ . We let  $\mathcal{O}_\varphi$  be the ring of integers at  $\varphi$ , i.e., the completion of the ring of rational functions defined at  $\varphi$ . Let  $V$  be a vector space over  $K$ . A coherent system of lattices in  $X$  is a family  $\{\Lambda_\varphi\}_{\varphi \in X}$  satisfying the following conditions ([18], Ch. VI, p.97):

1. Every  $\Lambda_\varphi$  is a  $\mathcal{O}_\varphi$ -lattice in  $V_\varphi$ .
2. There exists an affine set  $C \subset X$  and a lattice  $L$  over the ring  $\mathcal{O}_X(C)$ , of rational functions defined everywhere in  $C$ , such that  $L_\varphi = \Lambda_\varphi$  for every  $\varphi \in C$ .

Let  $V_{\mathbb{A}}$  be the adelization of the space  $V$ , and let us identify  $V$  with a discrete subgroup of  $V_{\mathbb{A}}$  as in [18]. Then for any coherent system  $\tilde{\Lambda} = \{\Lambda_\varphi\}_{\varphi \in X}$ , the product  $\Lambda_{\mathbb{A}} = \prod_{\varphi \in X} \Lambda_\varphi$  is an open and compact subgroup of  $V_{\mathbb{A}}$ , and every open and compact  $\mathcal{O}_{\mathbb{A}}$ -submodule of  $V_{\mathbb{A}}$  arises in this way ([18], §VI, Prop 1). For every affine subset  $C$  of  $X$  we define  $\Lambda_{\mathbb{A}, C} = \prod_{\varphi \in X \setminus C} V_\varphi \times \prod_{\varphi \in C} \Lambda_\varphi$ . Then  $\Lambda(C) = \Lambda_{\mathbb{A}, C} \cap V$  defines a sheaf  $\Lambda$  on  $X$ . We call a sheaf of this type an  $X$ -lattice. Equivalently, an  $X$ -lattice is a locally free sub-sheaf of  $V$ , where  $V$  is identified with the corresponding constant sheaf. Thus defined,  $X$ -lattices share some of the properties of usual lattices, namely:

- An  $X$ -lattice  $\Lambda$  is completely determined by the coherent system  $\{\Lambda_\varphi\}_\varphi$ .

- A coherent system can be modified at a finite number of places to define a new  $X$ -lattice. In particular,  $X$ -lattices can be defined by gluing together lattices defined over an affine cover.
- The adelization  $\mathrm{GL}_{\mathbb{A}}(V)$  of the general linear group  $\mathrm{GL}(V)$  of  $V$  acts on the set of lattices by acting on the family of compact and open  $\mathcal{O}_{\mathbb{A}}$ -submodules of  $V_{\mathbb{A}}$ .
- If  $V = \mathfrak{A}$  is an algebra, an  $X$ -lattice  $\mathfrak{D}$  is an order (i.e., a sheaf of orders) if and only if every completion is an order. The same holds for maximal orders.

For the proofs see [11] or [12]<sup>1</sup>. For any linear algebraic group acting on the space  $V$ , we have an induced action of the adelic group  $G_{\mathbb{A}}$  on the set of  $X$ -lattices in  $V$ . Two  $X$ -lattices are in the same  $G$ -genus if they are in the same orbit under this action. Similarly, classes are characterized as  $G_{K(V)}$ -orbits and spinor genera as  $G_{K(V)}\ker(\Theta)$ -orbits. It follows from our main theorem that there exist only a finite number of spinor genera in a genus for any group  $G$  satisfying **SN** and **RU**. On the contrary, the number of classes in a genus is frequently infinite, as in the examples we show in §4.

Note that  $X$ -lattices are locally free sheaves over the structure sheaf of  $X$ , and therefore they are associated to vector bundles [5]. The assumption made here that an  $X$ -lattice is contained in the constant sheaf  $V$  is not restrictive since for any locally free sheaf  $\Lambda$  on  $X$  the sheaf  $V = \Lambda \otimes_{\mathcal{O}_X} K$  is constant and a finite vector space over  $K$ . Furthermore, any isomorphism between two  $X$ -lattices in a space  $V$  can be extended to a linear map on  $V$ , whence next result follows:

**Proposition 5.** *The set of isomorphism classes of vector bundles of rank  $n$  over a curve  $X$  defined over a finite field is in correspondence with the set of double cosets  $\mathrm{GL}(K, n) \backslash \mathrm{GL}(\mathbb{A}_K, n) / \mathrm{GL}(\mathcal{O}_{\mathbb{A}}, n)$ .  $\square$*

It is not true in general that this set of double cosets is finite or that their elements are parameterized by their images under the reduced norm as it is the case for lattices over affine subsets. This follows easily from the classification results for vector bundles over arbitrary fields. See for example [5].

---

<sup>1</sup>Although [12] assumes characteristic 0 throughout, this hypotheses is not used for the results quoted here.



An  $X$ -lattice  $\Lambda$  in a space  $V$  is completely decomposable if  $\Lambda = \bigoplus_i J_i v_i$ , where  $\{v_1, \dots, v_n\}$  is a basis of the space  $V$  and  $J_1, \dots, J_n$  are  $X$ -lattices in  $K(X)$ . Note that every such lattice has the form  $J_i = \mathfrak{L}^{B_i}$ , where

$$\mathfrak{L}^B(C) = \left\{ f \in K(X) \mid \operatorname{div}(f)|_C \geq -B|_C \right\},$$

for some divisor  $B$  on  $X$ . Not every  $X$ -lattice is completely decomposable, as follows from the corresponding result for vector bundles [5]. In next section we need the following result:

**Lemma 6.** *There is a correspondence between conjugacy classes of maximal  $X$ -orders in  $\mathbb{M}_n(K)$  and isomorphism classes of  $n$ -dimensional vector bundles over  $X$  up to multiplication by invertible bundles.*

*Proof.* Since all maximal orders are locally conjugate at all places, any maximal  $X$ -order on  $\mathfrak{A}$  has the form  $b\mathfrak{D}_0 b^{-1}$  where  $b \in \mathfrak{A}_{\mathbb{A}}$  is a matrix with adelic coefficients and  $\mathfrak{D}_0 \cong \mathbb{M}_n(\mathcal{O}_X)$  is the sheaf of matrices with regular coefficients. We know that the adelization  $\mathfrak{D}_{0\mathbb{A}}$  is the ring of all adelic matrices  $c$  satisfying  $c(\mathcal{O}_X^n) = \mathcal{O}_X^n$ . It follows that  $b\mathfrak{D}_0 b^{-1}$  is the ring of all adelic matrices  $c$  satisfying  $c\Lambda = \Lambda$  where  $\Lambda = b\Lambda_0 = b(\mathcal{O}_X^n)$ . Since the stabilizer of a local order  $\mathfrak{D}_{\wp}$  is  $\mathfrak{D}_{\wp}^* K_{\wp}^*$ , it follows that two  $X$ -lattices  $\Lambda_1$  and  $\Lambda_2$  corresponds to the same maximal order, if and only if  $\Lambda_1 = d\Lambda_2$  for some  $d$  in the group  $J_X$  of ideles on  $X$ . The result follows since the idele  $d$  generates the invertible bundle  $\mathfrak{L}^{-\operatorname{div}(d)}$ .  $\square$

Note that split orders correspond to completely decomposable bundles. In particular, not all maximal orders are split.

**Proof of Theorem 1** The set of spinor genera in a genus is in one to one correspondence with the Abelian group

$$G_{\mathbb{A}}/G_{\mathbb{A}}^{\Lambda} G_{K(V)} \ker(\Theta). \quad (1)$$

If the group  $G$  satisfies condition **RU**, then  $H(K_{\wp}, F) = K_{\wp}^*/K_{\wp}^{*n}$ . By Lemma 2, the image of  $\Theta$  is contained in  $J_X/J_X^n$ , where  $J_X$  is the idele group of  $K(X)$ . Since the spinor norm is surjective in both  $K(X)$  and  $K_{\wp}$  (Lemma 1), the group (1) is isomorphic to  $J_X/K(X)^*H(\Lambda)$ , where  $H(\Lambda)$  is the pre-image in  $J_X$  of the group  $\Theta(G_{\mathbb{A}}^{\Lambda}) \subseteq J_X/J_X^n$ . We let  $\Sigma_{\Lambda}^C$  be the class field associated to the open subgroup  $K(X)^*H(\Lambda)$  ([18], §XIII.9). The set of  $G$ -spinor genera

in the  $G$ -genus of  $\Lambda$  is a principal homogeneous space, via Artin Map, for the group  $\mathcal{G} = \text{Gal}(\Sigma_\Lambda^C/K(X))$ . The element of  $\mathcal{G}$  sending the spinor genus of an  $X$ -lattice  $M$  to the spinor genus of a second  $X$ -lattice  $M'$  is defined by  $\rho(M, M') = [a, \Sigma_\Lambda/K(X)]$ , where  $x \mapsto [x, \Sigma_\Lambda/K(X)]$  denotes the Artin map, and  $a$  is any element of  $J_X$  satisfying  $\Theta(g) = aJ_X^n$  for some  $g \in G_\mathbb{A}$  such that  $M' = g(M)$ . The last statement follows from the identity

$$H(L) = H(\Lambda) \left( \prod_{\wp \in C \setminus D} \theta_\wp(G_\wp) \right) = H(\Lambda) \left( \prod_{\wp \in C \setminus D} K_\wp^* \right),$$

which follows from condition **SN** and the surjectivity of  $\theta_\wp$ .  $\square$

**Example A (continued).** Let  $\Lambda$  be the free  $X$ -lattice with basis  $\{v_i\}_{i=1}^n$  in the quadratic space described in the introduction. Then  $\Sigma_\Lambda = \mathbb{L}(t)$ , where  $\mathbb{L}$  is the only quadratic extension of  $\mathbb{F}$ , by a straightforward local computation. To find a representative in every spinor genus we observe that the adelic orthogonal element  $g = a(\lambda)$  defined by  $g_\wp(v_i) = v_i$  for  $i = 1, \dots, n-2$ ,  $g_\wp(v_{n-1}) = \lambda_\wp v_{n-1}$ , and  $g_\wp(v_n) = \lambda_\wp^{-1} v_n$  has spinor norm  $\lambda = (\lambda_\wp)_\wp$ . Now take  $u \in J_X$ , and let  $B = \text{div}(u)$  be the corresponding divisor. To find a representative  $L_u$  of the spinor genus corresponding to the class of  $u$  in  $J_X/J_X^2 H(\Lambda)$  we set  $\lambda = u$  above, whence

$$L_u = a(u)\Lambda = \bigoplus_{i=1}^{n-2} \mathcal{O}_X v_i \perp (\mathfrak{L}^B v_{n-1} \oplus \mathfrak{L}^{-B} v_n).$$

Note that  $L_u = L(B)$  depends only on the divisor  $B$  of  $u$ .

**Proposition 7.** *Assume that  $C$  is a (necessarily proper) open set such that  $G_{X \setminus C}$  is non-compact. Then two  $C$ -lattices  $N$  and  $M$  in the same  $G$ -genus are in the same  $G$ -class if and only if  $\rho(N, M)$  is the trivial element in  $\text{Gal}(\Sigma_N^C/K(X))$ .*

*Proof.* By the strong approximation theorem over function fields [13], the universal cover  $\tilde{G}$  of  $G$  has the strong approximation property with respect to the set  $S = X \setminus C$ . Assume that  $\rho(N, M)$  is trivial. Then  $N$  and  $M$  are in the same spinor genus, i.e., there exist  $g \in G$  and  $h \in \ker \Theta$  such that  $gh(N) = M$ . Then any pre-image  $\tilde{h}$  of  $h$  can be arbitrarily approximated by an element  $\tilde{f}$  in  $\tilde{G}$  whose image  $f \in G$  approximates  $h$ . Since lattice stabilizers in  $G_\mathbb{A}$  are open, the result follows.  $\square$

Note that in example **A**, the group  $G$  has strong approximation with respect to every non-empty finite subset of  $X$ . It follows that two lattices are in the same spinor genus if and only if they are conjugate over every affine subset of  $X$ .

**Corollary 7.1.** *Let  $\mathfrak{D}$  and  $\mathfrak{D}'$  be two maximal  $C$ -orders in the central simple algebra  $\mathfrak{A}$  that is not totally ramified at one or more infinite places of  $C$ . Then  $\mathfrak{D}$  and  $\mathfrak{D}'$  are conjugate if and only if  $\rho(\mathfrak{D}, \mathfrak{D}') = \text{id}$ .*

**Corollary 7.2.** *Assume  $\text{char}(\mathbb{F}) \neq 2$ . Let  $N$  and  $M$  be two  $C$ -lattices in the quadratic space  $W$  of dimension at least 3 (or a skew-hermitian space of rank at least 2) that belong to the same genus. Assume that  $W$  is isotropic at one or more infinite places of  $C$ . Then  $N$  and  $M$  are in the same  $G$ -class if and only if  $\rho(N, M) = \text{id}$ .*

Next corollary follows now from ([11], p.170) and ([16], p.363).

**Corollary 7.3.** *Let  $N$  and  $M$  be two  $C$ -lattices in the quadratic space  $W$  of dimension at least 5 (or the skew-hermitian space  $W$  of rank at least 4) that belong to the same genus. If  $C$  is any proper open subset of  $X$ , then  $N$  and  $M$  are in the same  $G$ -class if and only if  $\rho(N, M) = \text{id}$ .*

Recall that the space  $\lambda(X)$  of global sections of a lattice  $\lambda$  is a finite dimensional vector space over finite field  $\mathbb{F} = \mathcal{O}_X(X)$  ([18], Chapter VI). Furthermore, for any  $n$ -linear map  $\tau : V^n \rightarrow W$  satisfying  $\tau(\lambda^n) \subseteq M$ , where  $\lambda$  is a lattice in  $V$  and  $M$  is a lattice in the  $K(X)$ -vector space  $W$ , there exists an induced map  $\tilde{\tau} : \lambda(X)^n \rightarrow M(X)$  that is  $n$ -linear over  $\mathbb{F}$ . In particular:

1. If  $\lambda$  is an order, then  $\lambda(X)$  is a  $\mathbb{F}$ -algebra.
2. If  $\lambda$  is an integral quadratic lattice, then  $\lambda(X)$  is, naturally, a quadratic space over  $\mathbb{F}$ .

This observation is used throughout next section.

**Example: Maximal orders in  $\mathbb{M}_2(K)$ .** If  $n = 2$  every split maximal order has the form

$$\mathfrak{D}_B = \begin{pmatrix} \mathcal{O}_X & \mathfrak{L}^B \\ \mathfrak{L}^{-B} & \mathcal{O}_X \end{pmatrix},$$

where  $B$  is a divisor of  $X$  defined over  $\mathbb{F}$ . If  $B$  is a principal divisor, the ring of global sections  $\mathfrak{D}_B(X)$  is isomorphic to the matrix algebra  $\mathbb{M}_2(\mathbb{F})$ . If  $B$  is not principal, then  $\mathfrak{L}^B$  and  $\mathfrak{L}^{-B}$  cannot have a global section simultaneously. In fact, if  $\text{div}(f) + B \geq 0$  and  $\text{div}(g) - B \geq 0$  then  $B = \text{div}(g) = \text{div}(f^{-1})$ . We conclude that either  $\mathfrak{D}_B(X) \cong \mathbb{F} \times \mathbb{F}$  or  $\mathfrak{D}_B(X) \cong (\mathbb{F} \times \mathbb{F}) \oplus V$ , where  $V$  is an ideal of nilpotency degree 2. Note that the dimension of  $V$  tends to  $\infty$  with the degree of  $B$  by Riemann Roch's Theorem, whence we conclude that there exist infinite many conjugacy classes of maximal  $X$ -orders in  $\mathfrak{A}$ . In fact, we can give a more precise result:

**Proposition 8.** *The maximal orders  $\mathfrak{D}_B$  and  $\mathfrak{D}_D$  defined in the previous example are conjugate if and only if  $B$  is linearly equivalent to either  $D$  or  $-D$ .*

*Proof.* If  $B$  is principal, and if  $\mathfrak{D}_B$  is conjugate to  $\mathfrak{D}_D$ , we must have  $\mathfrak{D}_D(X) \cong \mathbb{M}_2(\mathbb{F})$ , and therefore  $D$  is principal. We can assume therefore that neither  $B$  nor  $D$  is principal. Replacing  $B$  or  $D$  by  $-B$  or  $-D$  if needed, we may assume  $B, D \geq 0$ . Let  $U$  be a global matrix such that  $\mathfrak{D}_B = U\mathfrak{D}_D U^{-1}$ . From the explicit description of  $\mathfrak{D}_B(X)$  given earlier, we conclude that the  $\mathbb{F}$ -vector spaces  $\mathfrak{L}^B(X)$  and  $\mathfrak{L}^D(X)$  have the same dimension. Furthermore, if  $W_B$  and  $W_D$  denote the  $K$ -vector spaces spanned by  $\mathfrak{D}_B(X)$  and  $\mathfrak{D}_D(X)$  respectively, then  $W_B = UW_D U^{-1}$ . There are two cases to be considered:

1. If  $\mathfrak{L}^B(X) \neq \{0\}$ , then  $W_B = W_D = KE_{1,1} \oplus KE_{2,2} \oplus KE_{1,2}$ .
2. If  $\mathfrak{L}^B(X) = \{0\}$ , then  $W_B = W_D = KE_{1,1} \oplus KE_{2,2}$ .

In the first case, we conclude that  $U$  has the form  $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ . In particular we must have

$$\mathfrak{L}^{-B} E_{2,1} = E_{2,2} \mathfrak{D}_B E_{1,1} = E_{2,2} (U \mathfrak{D}_D U^{-1}) E_{1,1} = a^{-1} c \mathfrak{L}^{-D} E_{2,1}.$$

We conclude that  $B = D + \text{div}(ac^{-1})$ , and therefore  $B$  and  $D$  are linearly equivalent. In the second case  $U$  has either the form  $\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$ , which is similar to the previous case, or the form  $\begin{pmatrix} 0 & a \\ c & 0 \end{pmatrix}$ , so that  $B = -D + \text{div}(ac^{-1})$ , and  $B$  is linearly equivalent to  $-D$ .  $\square$

## 4 Representation fields

Let  $\Lambda$  be an  $X$ -lattice in a  $K(X)$ -vector space  $V$  as before, and let  $M$  be an  $X$ -lattice in a subspace  $W \subseteq V$ . We allow the case  $V = W$ . Assume  $M \subseteq \Lambda$  in all that follows. Assume that  $G$  is a semi-simple linear algebraic sub-group of  $\mathrm{GL}(V)$  satisfying the conditions **SN** and **RU**. Following the notations in [1] we call an element  $u \in G_{\mathbb{A}}$  a generator for  $\Lambda|M$  if  $M \subseteq u\Lambda$ . Local generators are defined analogously. Note that  $u \in G_{\mathbb{A}}$  is a generator if and only if  $u_{\wp}$  is a local generator for every place  $\wp$ . As usual we say that  $M$  is  $G$ -represented by an  $X$ -lattice  $N$  in  $V$ , or that  $N$   $G$ -represents  $M$ , if  $N$  contains a lattice in the the  $G$ -orbit of  $M$ . We say that a set  $\Psi$  of lattices  $G$ -represents  $M$  if some element of  $\Psi$  does. In this setting, we have the following proposition, whose proof is transliteration of the one in the number field case [1], and therefore is omitted.

**Proposition 9.** *In the above notations, let  $\Lambda'$  be an  $X$ -lattice in the  $G$ -genus of  $\Lambda$ . The lattice  $M$  is  $G$ -represented by the spinor genus of  $\Lambda'$  if and only if there exists a generator  $u$  for  $\Lambda|M$  such that  $\rho(\Lambda, \Lambda') = [\Theta(u), \Sigma_{\Lambda}/K(X)]$ .*

□

We denote by  $H(\Lambda|M)$  the pre-image in  $J_X$  of the set of spinor norms  $\Theta(u)$  of all generators  $u$  for  $\Lambda|M$ . If  $K(X)^*H(\Lambda|M)$  is a group, the corresponding class field is called the representation field  $F(\Lambda|M)$  for  $\Lambda|M$ . The spinor genus of a lattice  $\Lambda'$  represents  $M$  if and only if  $\rho(\Lambda, \Lambda')$  is trivial on  $F(\Lambda|M)$ . The proof of the following fact is also completely analogous to the number field case ([10] and [3]):

**Proposition 10.** *The representation field always exist for lattices in quadratic or quaternionic skew-hermitian spaces.*

□

The corresponding result for orders in quaternion algebras follows from the case of quadratic forms just as in the number field case, but it cannot be extended to algebras of higher dimension [4].

It must be kept in mind, however, that the classification of the lattices in a genus into spinor genera is a much coarser invariant than in the number field case, as the number of classes in a genus is usually infinite. The following example illustrate this:

**Example B (continued).** Let  $X$  and  $\Lambda$  be as in §4, but assume  $n = 4$  and the field of definition of  $X$  is  $\mathbb{F} = \mathbb{F}_q$  with  $q = 4t + 3$ . In particular, the space of global sections of  $L(B)$  is

$$\Lambda(X) = \mathcal{O}_X(X)v_1 + \mathcal{O}_X(X)v_2 + \mathfrak{L}^B(X)v_3 + \mathfrak{L}^{-B}(X)v_4.$$

By Riemann-Rochs Theorem, the  $\mathbb{F}$ -dimension of  $\Lambda(X)$  tends to infinity with the degree of  $B$ . In particular there exists infinitely many classes of such lattices, while they belong to the same spinor genus as long as  $\deg(B)$  is even. We claim that none of these lattices represents  $M = \mathcal{O}_X v_3 + \mathcal{O}_X v_4$  when  $B$  is not principal. In fact, if  $B$  fails to be principal, then either  $\mathfrak{L}^B(X)$  or  $\mathfrak{L}^{-B}(X)$  has dimension 0, whence the space of global sections  $\Lambda(X) = [\mathcal{O}_X(X)v_1 + \mathcal{O}_X(X)v_2] \perp U$  where  $U = \mathfrak{L}^B(X)v_3 + \mathfrak{L}^{-B}(X)v_4$  is the radical, and  $Z = \mathcal{O}_X(X)v_1 + \mathcal{O}_X(X)v_2$  is anisotropic by the choice of  $q$ , while  $M(X)$  is a hyperbolic plane. We note however that the theory of representation by spinor genera tell us that half of the spinor genera in the genus of  $\Lambda$  must represent  $M$ . It is not hard to see that the image of the spinor norm in this case is  $H(\Lambda) = J_X^n \mathcal{O}_A^*$ . It follows that there are more than one spinor genus representing  $M$  whenever the torsion subgroup of the Picard group of  $X$  has even order. In this case there must exist classes in the genus of  $\Lambda$  that are not in the class of any of the lattices  $L(B)$ .

Recall that an order of maximal rank in a central simple algebra  $\mathfrak{A}_{K(V)}$  is said to be split if it represents the  $n$ -fold cartesian product  $\mathcal{O}_X \times \cdots \times \mathcal{O}_X$ . A maximal  $X$ -order  $\mathfrak{D}$  in  $\mathbb{M}_n(K)$  is split if and only if the corresponding vector bundle is a direct product of one dimensional vector bundles, i.e., it corresponds to an  $X$ -lattice of the type

$$\Lambda = \mathfrak{L}^{B_1} \times \cdots \times \mathfrak{L}^{B_n}.$$

Note that if the order of diagonal matrices  $\bigoplus_i \mathcal{O}_X E_{i,i}$  is contained in  $\mathfrak{D}$ , every diagonal matrix unit  $E_{i,i}$  is a global section of  $\mathfrak{D}$ , and therefore  $\mathfrak{D} = \sum_{i,j} J_{i,j} E_{i,j}$  for some invertible bundle  $J_{i,j} \subseteq K$ , and the same holds for the rings of global sections. A simple computation shows  $J_{i,j} \cong \mathfrak{L}^{B_i - B_j}$ . An argument similar to that in the previous example can be used to prove next result:

**Proposition 11.** *If  $N$  is the total number of spinor genera of maximal  $X$ -orders in the matrix algebra  $\mathbb{M}_n(K)$ , where  $K$  is the field of functions on a smooth projective curve  $X$  over a finite field, then at least  $\frac{N}{n} - 1$  spinor genera contain non-split  $X$ -orders.*

*Proof.* One particular example of split order is the maximal order  $\mathfrak{D}_B$  corresponding to the  $X$ -lattice  $\mathfrak{L}^B \times \mathcal{O}_X \times \cdots \times \mathcal{O}_X$ . We claim that

1. Every spinor genera of maximal  $X$ -orders contains the order  $\mathfrak{D}_B$  for some divisor  $B$ .
2. The maximal orders  $\mathfrak{D}_B$  and  $\mathfrak{D}_D$  are in the same spinor genus if and only if  $B$  and  $D + nC$  are linearly equivalent for some divisor  $C$ .

The stabilizer of the local maximal  $X$ -order  $\mathfrak{D}_\phi$  is  $K_\phi^* \mathfrak{D}_\phi^*$  and its set of norms is  $K_\phi^{*n} \mathcal{O}_\phi^*$ . We conclude that  $H(\mathfrak{D}) = J_K^n \mathcal{O}_\mathbb{A}^*$ . It follows that the class group  $J_K/K^*H(\mathfrak{D})$  is isomorphic to the divisor group of  $X$  modulo  $n$ -powers. Observe that  $\mathfrak{D}_D = u\mathfrak{D}_B u^{-1}$  where  $u = \text{diag}(b, 1, \dots, 1)$ , and the idele  $b = n(u) \in J_K$  satisfies  $\text{div}(b) = D - B$ . It follows that  $\mathfrak{D}_D$  and  $\mathfrak{D}_B$  are in the same spinor genus if and only if  $D - B$  is 0 modulo  $n$ -powers in the divisor group of  $X$ . Furthermore, any spinor genera can be obtained in this way for an appropriate choice of  $b$  in  $J_X/K(X)^*H(\mathfrak{D})$ . In particular, every spinor genus contains a split order.

It follows from the previous argument that the class modulo  $n$  of the divisor  $B$  depends only on the spinor genera of the maximal order  $\mathfrak{D}_B$ . In particular, the degree of  $B$  is well defined for a particular spinor genus as an element of  $\mathbb{Z}/n\mathbb{Z}$ . We use this in all that follows.

Let  $\mathbb{L}$  be the only field extension of the finite field  $\mathbb{F}$  of degree  $n$ . We claim that, if  $\mathbb{L}$  embeds into  $\mathfrak{D}(X)$  for a split order  $\mathfrak{D}$ , then  $\mathfrak{D} \cong \mathbb{M}_n(\mathcal{O}_X)$ . In fact, let  $\Lambda = \mathfrak{L}^{B_1} \times \cdots \times \mathfrak{L}^{B_n}$  be the lattice corresponding to  $\mathfrak{D}$ . Then  $\mathfrak{D} = (\mathfrak{L}^{B_i - B_j})_{i,j}$ . We define an order in the group of divisor classes by  $D \preceq C$  if  $D \leq C + \text{div}(f)$  for some  $f \in K$ . Note that  $\mathfrak{L}^{C-D}$  has a non-trivial global section if and only if  $D \preceq C$ . We assume that the  $B_i$ 's have been re-arranged in a way that  $B_n$  is minimal with respect to this order, and  $B_{r+1}, \dots, B_n$  are all the divisors that are linearly equivalent to  $B_n$ . Then any global section of  $\mathfrak{D}$  has the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where  $A$  is an  $r$ -times- $r$  block. It follows that  $K\mathbb{L}$  has a representation of dimension  $r < n$  over  $K$ , and therefore  $r = 0$ .

Now, the proposition follows if we prove that for any divisor  $B$  such that  $\deg B \equiv 0 \pmod{n}$ , there exists a maximal  $X$ -order  $\mathfrak{D}$  in the same spinor genus as  $\mathfrak{D}_B$  for which there is an embedding  $\mathbb{L} \hookrightarrow \mathfrak{D}(X)$ , since we know that  $\mathfrak{D}$  cannot be a split order unless  $B$  is principal.

To prove this we let  $L = K\mathbb{L} = \mathbb{L} \otimes_{\mathbb{F}} K$ , and let  $\mathfrak{H} = \mathbb{L} \otimes_{\mathbb{F}} \mathcal{O}_X$  be the only maximal order in the  $K$ -algebra  $L$ . Note that if  $Y$  is the projective curve over  $\mathbb{L}$  defined by the same equations defining  $X$  over  $\mathbb{F}$ , and  $\phi : Y \rightarrow X$  is the natural morphism of schemes, then  $\mathfrak{H}$  is the push-forward to  $X$  of the structure sheaf on  $Y$ . In particular  $\mathfrak{H}(X) = \mathbb{L}$ .

Consider the natural embedding

$$\phi : \mathfrak{H} = \mathbb{L} \otimes_{\mathbb{F}} \mathcal{O}_X \hookrightarrow \mathbb{M}_2(\mathbb{F}) \otimes_{\mathbb{F}} \mathcal{O}_X = \mathfrak{D}_0$$

induced by an arbitrary embedding  $\mathbb{L} \hookrightarrow \mathbb{M}_2(\mathbb{F})$ . Then the order  $\mathfrak{H}' = \phi(\mathfrak{H})$  is contained in some maximal order in the spinor genera of  $\mathfrak{D}_B$  if and only if we can write  $\mathfrak{D}_B = u\mathfrak{D}_0u^{-1}$  where the image of the reduced norm  $n(u)$  in the quotient  $J_K/K^*H(\mathfrak{D})$  coincide with the image of a generator. Note that if we identify  $L$  with the sub-algebra of  $\mathfrak{A}$  spanned by  $\mathfrak{H}'$ , the group of invertible elements  $L_{\mathbb{A}}^*$  (all of which are generators for  $\mathfrak{D}|\mathfrak{H}'$ ) is isomorphic to the group of ideles  $J_L$  of  $L$ , and the reduced norm  $n : J_L \rightarrow J_K$  is just the field norm  $n_{L/K}$ . It follows from Theorem 7 in chapter XIII of [18], that  $H_L = K^*n_{L/K}(J_L)$  is the kernel of the Artin map  $t \mapsto [t, L/K]$  on ideles. In particular  $H_L$  has index at most  $n$  in  $J_K$ , and we can check that the divisor of every idele in  $H_L$  has degree in  $n\mathbb{Z}$  by computing the degrees of the generators. We conclude that  $H_L$  is the group of all ideles whose divisors have degrees in  $n\mathbb{Z}$ , whence the result follows.  $\square$

**Remark 1.** Since for every curve  $\text{Pic}(X) \cong \mathbb{Z} \times T$  where  $T \cong \text{Pic}^0(X)$  is a finite group ([18], §IV.4, Theorem 7), we conclude that  $J_K/K^*H(\mathfrak{D}) \cong (\mathbb{Z}/n\mathbb{Z}) \times (T/nT)$ . In particular, the bound in the proposition is  $|T/nT| - 1$ .

**Remark 2.** Assume for simplicity that  $\mathbb{K}$  has odd characteristic and  $n = 2$ . Let  $B = \text{div}(b)$  be a divisor of even degree, with  $b \in J_K$ . An order  $\mathfrak{D}$  in the same spinor genera as  $\mathfrak{D}_B$ , representing the maximal order of  $L$ , is given as follows: Tchebotarev Density Theorem ([15], Thm. 9.13A) implies the existence of a place  $\wp \in X$ , such that any idele  $j$ , where  $j_{\wp}$  is a uniformizer of  $K_{\wp}$ , and  $j_{\mathfrak{q}} = 1$  if  $\mathfrak{q} \neq \wp$ , satisfies  $bj^{-1} \in K^*H(\mathfrak{D})$ . Note that  $\wp$  has even degree, whence the field  $\mathbb{L}$  embeds into  $K_{\wp}$ . We may assume  $\mathbb{L} = \mathbb{F}(u)$ , where  $u$  is a root of  $x^2 = \delta$  for some  $\delta \in \mathbb{F}$ . Then  $\mathbb{L}K$  embeds into  $\mathbb{M}_2(K)$  by sending  $u$  to the matrix  $\begin{pmatrix} 0 & 1 \\ \delta & 0 \end{pmatrix}$ . Let  $P$  be the divisor corresponding to  $\wp$ . Then we may choose  $\mathfrak{D} = b\mathfrak{D}_0b^{-1}$  where  $b_{\mathfrak{q}}$  is the identity matrix for  $\mathfrak{q} \neq \wp$  and

$$b_{\wp} = A \begin{pmatrix} j_{\wp} & 0 \\ 0 & 1 \end{pmatrix} A^{-1}, \text{ where } A \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix} A^{-1} = \begin{pmatrix} 0 & 1 \\ \delta & 0 \end{pmatrix}.$$



Note that  $\det b = j$ .

**Example 1.** When  $X = \mathbb{P}^1$  is the projective plane, the lower bound given by this result is 0, so in principle there could be no non-split  $X$ -orders in  $\mathbb{M}_n(K)$ . In fact, the Birkhoff-Grothendieck Theorem<sup>2</sup> implies that every such  $X$ -lattice is a sum of  $X$ -lattices of rank 1. It follows that non-split orders fail to exist and the bound is sharp in this case.

**Example 2.** Consider the plane curve  $X$  of genus 1 with projective equation  $y^2z - x(x^2 - z^2) = 0$  over a finite field of odd characteristic. Then  $\operatorname{div}(x) = 2(P_0 - P_\infty)$ , while there is no element in  $K = K(X)$  whose divisor is  $P_0 - P_\infty$ , or such element would generate the field  $K$ . We conclude that  $\operatorname{Pic}^0(X)$  has an element of order 2, and therefore its order is even. We conclude the existence of at least one class of non-split maximal orders in  $\mathbb{M}_2(K)$ , or equivalently, a non-split vector bundle defined over  $\mathbb{F}$ .

## References

- [1] L. ARENAS-CARMONA, Applications of spinor class fields: embeddings of orders and quaternionic lattices, *Ann. Inst. Fourier*, 53 (2003), 2021-2038.
- [2] L. ARENAS-CARMONA, Relative spinor class fields: A counterexample, *Archiv. Math.* **91** (2008), 486-491.
- [3] L. ARENAS-CARMONA, Representation fields for skew-hermitian forms, *Archiv. Math.* **94** (2010), 351-356.
- [4] L. ARENAS-CARMONA, Representation fields for commutative orders, To appear in *Ann. Inst. Fourier*.
- [5] M. ATIYAH, Vector bundles over an elliptic curve, *Proc. London Math. Soc.*, 7 (1957), 414-452.
- [6] L.BODNARCHUK, I.BURBAN, YU.DROZD, G.-M.GREUEL, Vector bundles and torsion free sheaves on degenerations of elliptic curves, in *Global Aspects of Complex Geometry*, 83-129, Springer, (2006), *math.AG/0603261*.
- [7] S. BÖGE, Spinorgeschlechter schiefhermitescher Formen, *Arch. Math.*, XXI (1970), 172-184.
- [8] J. BRZEZINSKI, Riemann-Roch Theorem for locally principal orders, *Math. Ann.*, 276 (1987), 529-536.
- [9] C. CHEVALLEY, *L'Arithmétique dans les algèbres de matrices*, Hermann, Paris, 1936.
- [10] J.S. HSIA, Y.Y. SHAO, F. XU, Representation of indefinite quadratic forms, *J. reine angew. Math.*, **494** (1998), 129-140.
- [11] O.T. O'MEARA, *Introduction to Quadratic Forms*, Spriger-Verlag, Berlin, 1973.

---

<sup>2</sup> A proof of this result for arbitrary fields is given in [6] (Theorem 2.1)

- [12] V. PLATONOV AND A. RAPINCHUK, Algebraic groups and number theory, Academic Press, Boston, 1994.
- [13] G. PRASAD, Strong approximation for semi-simple groups over function fields, *Annals of Mathematics*, 105 (1977), 553-572.
- [14] I. REINER, Maximal Orders, Academic Press, London, 1975.
- [15] M. ROSEN, *Number Theory in Function Fields*, Springer Verlag, New York, 2002.
- [16] W. SCHARLAU, Quadratic and Hermitian Forms, Springer-Verlag, Berlin, 1985.
- [17] J.-P. SERRE, Cohomologie galoisienne: progrès et problèmes, Séminaire Bourbaki, Vol. 1993/94, *Astérisque*, 227 (1995), Exp.No. 783, 4, 229-257.
- [18] A. WEIL, Basic Number Theory, Second ed., Springer-Verlag, Berlin, 1973.